Stochastic Portfolio Theory Optimization and the Origin of Rule-Based Investing.

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Modern Portfolio Theory (MPT) «Cornerstones»

Markowitz Mean-Variance Optimization (Markowitz, 1952):
- Choose allocation maximizing portfolio expected returns at given level of risk (or minimizing portfolio risk at fixed expected return)
- Normative theory, based on the assumption that investors use quadratic utility function

Market Portfolio as Equilibrium Portfolio
- Consequence of Mutual Fund Separation Theorem (Tobin, 1958)
  - Market portfolio = most efficient portfolio
  - Origin of cap-weighted benchmarks

Capital Asset Pricing Model (CAPM – Sharpe 1964, Lintner 1965, Mossin 1966)
- At equilibrium asset risk premiums are proportional to market risk premium via the «beta» measure of systematic risk
Rule-Based (or «Smart Beta») Investing: a Brief Review

- Risk-focussed approaches
  - Minimum Variance (Haugen & Baker, 1991)
  - Risk Parity (Qian, 2005 – Roncalli, Maillard & Teiletche, 2010)
  - Maximum Diversification (Choueifaty & Coignard, 2008)

- Agnostic approaches
  - Equal Weighting (DeMiguel, Garlappi and Uppal, 2009)

- Fundamental-focussed approaches
  - High Dividend Yield (Arnott, Hsu and Moore, 2005)
  - RAFI Methodology
**Rule-Based Investing: Evidence**

Most rule-based non cap-weighted indices out perform in the long term.

<table>
<thead>
<tr>
<th>Index</th>
<th>Factor Exposures</th>
<th>Total Return</th>
<th>Total Risk</th>
<th>Active Return</th>
<th>Active Risk</th>
<th>Annual Turnover</th>
<th>Pairwise Correlation</th>
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</thead>
<tbody>
<tr>
<td>MSCI World</td>
<td>--</td>
<td>7.1</td>
<td>15.4</td>
<td>0.0</td>
<td>0.0</td>
<td>3.9</td>
<td>NA</td>
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<tr>
<td>MSCI World Equal Weighted</td>
<td>Size</td>
<td>8.3</td>
<td>16.3</td>
<td>1.2</td>
<td>5.2</td>
<td>31.8</td>
<td>0.22</td>
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<tr>
<td>MSCI World Minimum Volatility</td>
<td>Volatility</td>
<td>8.5</td>
<td>11.6</td>
<td>1.4</td>
<td>6.7</td>
<td>20.0</td>
<td>0.30</td>
</tr>
<tr>
<td>MSCI World Value Weighted</td>
<td>Value</td>
<td>8.6</td>
<td>15.6</td>
<td>1.5</td>
<td>3.6</td>
<td>20.3</td>
<td>0.30</td>
</tr>
<tr>
<td>MSCI World Risk Weighted</td>
<td>Size, Volatility</td>
<td>9.5</td>
<td>13.7</td>
<td>2.4</td>
<td>5.3</td>
<td>27.2</td>
<td>0.46</td>
</tr>
<tr>
<td>MSCI World Quality</td>
<td>Growth, Leverage</td>
<td>10.9</td>
<td>14.0</td>
<td>3.8</td>
<td>5.9</td>
<td>27.6</td>
<td>0.13</td>
</tr>
<tr>
<td>MSCI World Momentum</td>
<td>Momentum</td>
<td>10.4</td>
<td>15.9</td>
<td>3.3</td>
<td>8.5</td>
<td>127.5</td>
<td>0.03</td>
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<tr>
<td>MSCI World HDY</td>
<td>--</td>
<td>10.3</td>
<td>14.6</td>
<td>3.2</td>
<td>6.5</td>
<td>22.0</td>
<td>0.41</td>
</tr>
</tbody>
</table>

Source: Equity Index Handbook, UBS & MSCI, 2014

What about transporting performance coming from equity alternative risk factors while remaining market neutral?
Rule-Based Investing: Evidence and Open Issues

**Evidence 1**: non cap-weighted allocation strategies tend to outperform their cap-weighted counterparts because of exposure to alternative risk factors, such as value, size and low volatility (Chow, Hsu, Kalesnik, and Little, 2011)

**Evidence 2**: investing in just one type of rule-based methodology may lead to unwanted concentration and cluster risks. Possible solution: diversify across different rule-based, non cap-weighted allocation methods (Gander, Leveau and Pfiffner, 2013)

**Open Issue 1**: what is the reason why any rule-based allocation, including inverse approaches, seems to outperform cap-weighted benchmarks (in the long term)?

**Open Issue 2**: missing theoretical justification for most rule-based allocation approaches (apart from minimum variance portfolio)

- Mean-variance (and minimum variance) portfolios can be derived from utility maximization problem
- Rule-based allocation approaches often introduced based on heuristic arguments/“sensible principles”
- Missing bridge between MPT and rule-based strategies
Agenda

Markowitz’s Mean-Variance Optimization Revisited
- Seeds of some rule-based strategies are present in Markowitz’s optimization solution
- Optimality of rule-based strategies implies expected return assumptions

Connection between Stochastic Portfolio Theory (SPT) and Rule-Based Investing
- SPT-driven expected returns and related optimal portfolio selection results
  - SPT as a tool to derive a unified optimization theory, accounting for both Markowitz’s mean-variance portfolio solution and for a blend of rule-based allocations
- SPT explanation of rule-based portfolios’ long term outperformance vs. cap-weighted benchmarks
- Relative optimization problem in SPT: analytical results and open issues
  - SPT-optimal portfolio as a combination of rule-based approaches

Conclusions
Markowitz’s Mean-Variance Optimization Revisited
Markowitz’s Utility Maximation Problem

- Portfolio weight vector \( \pi(t) \equiv (\pi_1(t), \cdots, \pi_n(t)) \)
- Introduce vector \( e \) with all entries equal to 1
- Maximize utility function with budget and volatility constraints:

\[
U(\pi(t), ER(t), \sigma(t)) = \pi(t)^T ER(t) - \lambda_1 \left( \pi^T(t)e - 1 \right) - \\
- \lambda_2 \left[ \pi(t)^T \sigma(t) \pi(t) - \sigma_B^2 \right].
\]

- Optimal portfolio solution:

\[
\pi(t) = A_1(t) \frac{\sigma^{-1}(t)ER(t)}{e^T \sigma^{-1}(t) ER(t)} + [1 - A_1(t)] \frac{\sigma^{-1}(t)e}{e^T \sigma^{-1}(t)e}
\]

- Maximum Sharpe Portfolio
- Global Minimum Variance Portfolio
Analytical Expression for the Inverse Covariance Matrix

Objective: analytically study the term $\sigma^{-1}(t)ER(t)$

Covariance matrix = Hadamard (element by element) product of correlation matrix and volatility-dependent matrix

$$\sigma(t) = \Gamma(t) \circ R(t) = \Gamma(t) \circ (R_0(t) + \Delta R(t))$$

Volatility-dependent matrix

$$\Gamma_{ij}(t) = \sigma_i(t)\sigma_j(t)$$

$R_0(t)$ equal correlation matrix

$\Delta R(t)$ difference between true and equal correlation matrix
Correlation Matrix Decomposition

Equal correlation matrix

\[
R_0(t) = \begin{pmatrix}
1 & \bar{\rho}(t) & \bar{\rho}(t) & \cdots & \bar{\rho}(t) \\
\bar{\rho}(t) & 1 & \bar{\rho}(t) & \cdots & \bar{\rho}(t) \\
\bar{\rho}(t) & \bar{\rho}(t) & 1 & \cdots & \bar{\rho}(t) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\bar{\rho}(t) & \bar{\rho}(t) & \bar{\rho}(t) & \cdots & 1
\end{pmatrix}
\]

\[
\bar{\rho}(t) = \frac{1}{N(N-1)} \sum_{i \neq j}^{N} \rho_{ij}(t)
\]

Difference between true and equal correlation matrix

\[
\Delta R(t) = \begin{pmatrix}
0 & \rho_{12}(t) - \bar{\rho}(t) & \cdots & \rho_{1n}(t) - \bar{\rho}(t) \\
\rho_{21}(t) - \bar{\rho}(t) & 0 & \cdots & \rho_{2n}(t) - \bar{\rho}(t) \\
\rho_{31}(t) - \bar{\rho}(t) & \rho_{32}(t) - \bar{\rho}(t) & \cdots & \rho_{3n}(t) - \bar{\rho}(t) \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{n1}(t) - \bar{\rho}(t) & \rho_{n2}(t) - \bar{\rho}(t) & \cdots & 0
\end{pmatrix}
\]
Correlation Matrix Inversion

Inverse of correlation matrix

\[
[R_0(t) + \Delta R(t)]^{-1} = R_0^{-1}(t) - [1 + R_0^{-1}(t)\Delta R(t)]^{-1} R_0^{-1}(t)\Delta R(t) R_0^{-1}(t)
\]

Define «correction matrix»

\[
\Xi (R_0(t), \Delta R(t)) = [1 + R_0^{-1}(t)\Delta R(t)]^{-1} R_0^{-1}(t)\Delta R(t)
\]

Inverse of correlation matrix = linear operator acting on inverse of equal correlation matrix

\[
[R_0(t) + \Delta R(t)]^{-1} = \left\{1 - [1 + R_0^{-1}(t)\Delta R(t)]^{-1} R_0^{-1}(t)\Delta R(t)\right\} R_0^{-1}(t)
\]

\[
= \left\{1 - \Xi (R_0(t), \Delta R(t))\right\} R_0^{-1}(t)
\]
Write equal correlation matrix as:

$$R_0(t) = \rho(t)ee^T + (1 - \rho(t))1$$

Inverse equal correlation matrix final form

$$R_0^{-1}(t) = -\frac{\rho(t)}{(1 - \rho(t)) \left[ N\rho(t) + (1 - \rho(t)) \right]} ee^T + \frac{1}{1 - \rho(t)}1$$

Define coefficients

$$\psi(t) = \frac{1}{1 - \rho(t)}$$

$$\phi(t) = -\frac{\rho(t)}{(1 - \rho(t)) \left[ N\rho(t) + (1 - \rho(t)) \right]}$$
Inverse Covariance Matrix Derivation

Inverse covariance matrix = Hadamard (element by element) product of inverse volatility-dependent matrix and inverse correlation matrix

\[ \sigma^{-1}(t) = \tilde{\Gamma}(t) \circ (R_0(t) + \Delta R(t))^{-1} \]

\[ = \tilde{\Gamma}(t) \circ R_0^{-1}(t) - \tilde{\Gamma}(t) \circ \left[ \Xi(R_0(t), \Delta R(t)) R_0^{-1}(t) \right] \]

Inverse volatility-dependent matrix

\[ \tilde{\Gamma}_{ij}(t) = \frac{1}{\sigma_i(t) \sigma_j(t)} \]
Analytical Form of Markowitz’s Optimal Portfolio Solution

\[ \pi(t) = \tilde{A}_1(t) \left\{ \left[ 1 - \mathbb{E} \left(R_0(t), \Delta R(t) \right) \right] \left[ \psi(t)\pi_1(t) + \phi(t)\pi_{RP}(t) \right] \right\} + \\
+ (1 - \tilde{A}_1(t)) \frac{\sigma^{-1}(t)e}{e^T \sigma^{-1}(t)e} \]

- The generic solution to unconstrained portfolio optimization contains two rule-based allocations, the global minimum variance portfolio and the inverse-volatility portfolio.
- In order for the optimal solution to coincide exactly with one of the rule-based allocations, we need assumptions on expected returns.
- The statement according to which rule-based allocations avoid the problem of formulating return expectations is a misconception.
- In the two-asset case, \( \Delta R(t) = 0 \rightarrow \mathbb{E} \left(R_0(t), \Delta R(t) \right) = 0 \) the solution simplifies (no “correction” matrix needed).
Building Blocks of Optimal Portfolio Solution

Identity 1: \[(\tilde{\Gamma}(t) \circ 1) \cdot ER(t) = \begin{pmatrix}
\frac{1}{\sigma_1(t)} & \frac{ER_1(t)}{\sigma_1(t)} \\
\frac{1}{\sigma_2(t)} & \frac{ER_2(t)}{\sigma_2(t)} \\
\vdots \\
\frac{1}{\sigma_N(t)} & \frac{ER_N(t)}{\sigma_N(t)}
\end{pmatrix} = \pi_1(t)\]

Identity 2: \[(\tilde{\Gamma}(t) \circ ee^T) \cdot ER(t) = \sum_{i=1}^{N} \frac{ER_i(t)}{\sigma_i(t)} \begin{pmatrix}
\frac{1}{\sigma_1(t)} \\
\frac{1}{\sigma_2(t)} \\
\vdots \\
\frac{1}{\sigma_N(t)}
\end{pmatrix} = \pi_{RP}(t)\]

Inverse-volatility portfolio = variant of risk-parity portfolio when assuming equal correlation matrix
## Conditions for Optimality of Rule-Based Strategies

<table>
<thead>
<tr>
<th>ER(t)</th>
<th>( \pi_1(t) )</th>
<th>Optimal Portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e )</td>
<td>Inverse Variance</td>
<td>GMV = Inverse Variance + Inverse Volatility</td>
</tr>
<tr>
<td>diag(( \sigma(t) ))</td>
<td>EW</td>
<td>EW + Inverse Volatility + GMV (= Inverse Variance + Inverse Volatility)</td>
</tr>
<tr>
<td>( \bar{\sigma}(t) )</td>
<td>Inverse Volatility</td>
<td>Inverse Volatility + GMV (= Inverse Variance + Inverse Volatility)</td>
</tr>
</tbody>
</table>

Markowitz’s optimal portfolio solution contains rule-based building blocks:

- **Natural rule-based building blocks** (always part of the optimization problem solution): inverse volatility (new result) & global minimum variance (known result)
- Other rule-based building blocks need assumptions on expected returns
Stochastic Portfolio Theory (SPT): An Introduction
Stochastic Portfolio Theory (SPT): what is it?

- Theory first established by Robert Fernholz
- Applied since 1987 in mathematical investment programs
- Descriptive (as opposed to normative) theory of financial markets
- Consistent with either equilibrium or disequilibrium, with either arbitrage or no-arbitrage
- Allows analysis of long-term portfolio behavior
- Accounts for impact of compounding and rebalancing, often neglected in literature
- Provides framework for mathematical market models
SPT Assumptions

- Closed universe of constituents
- No transaction costs/taxes
- No problems with indivisibility of shares
  - One single share outstanding for each asset
  - Asset price = asset market cap
- Asset log-returns follow Brownian motion
- Market diversity: no single asset dominates the market in finite time
  - Common sense for a universe of asset classes/asset class partitions
  - Enforced in anti-trust laws
  - Consequence: growth rate of largest market cap lower than the market’s growth rate at some point
SPT Model for Asset Total Logarithmic Returns

- Asset total logarithmic returns

\[ d \log \hat{X}_i(t) = \gamma_i(t) dt + \delta_i(t) dt + \xi_i^T(t) dW(t) \]

- \( \hat{X}_i(t) \) “augmented” asset price (accounts for dividends/coupons)
- \( \gamma_i(t) \) logarithmic price growth rate
- \( \delta_i(t) \) cash flow rate of return, related to dividend or coupon payments
- \( \xi_i(t) \) \( k \)-dimensional vector, which measures the sensitivity of asset \( i \) to \( k \) Brownian diffusion processes \( W \)
SPT Model for Asset Total Linear Returns: Ito’s Lemma

Simple consequence of Ito’s Lemma

\[
\frac{d \hat{X}_i(t)}{\hat{X}_i(t)} = \left( \gamma_i(t) + \delta_i(t) + \frac{\sigma_{ii}(t)}{2} \right) dt + \xi_i^T(t) dW(t)
\]

Expected asset returns depend on

- Expected logarithmic price growth rates (hard to predict and volatile)
- Expected asset variances
- Expected contribution to returns from coupons and dividends
SPT Model for Asset Total Linear Returns: Evidence (1)

Contributions to S&P 500 Annual Total Linear Returns

- Log Price Return
- Dividend Contribution to Return
- Ito term contribution to return
- S&P 500 Annual Total Linear (Simple) Returns
The Connection between SPT and Rule-Based Investing:
SPT-driven Expected Returns and Optimal Portfolio Selection
Markowitz Optimal Portfolios with SPT-driven Expected Returns

Define portfolio value $Z_\pi(t)$

Portfolio linear return

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \sum_{i=1}^{n} \pi_i(t) \left[ \gamma_i(t) + \delta_i(t) + \frac{\sigma_{ii}(t)}{2} \right] dt + \sum_{i=1}^{n} \pi_i(t) \xi_i^T(t) dW(t)$$

Optimal portfolio solution

$$\pi(t) = \left[ A_1(t) \left( e^T \sigma^{-1}(t) \gamma(t) \right) \right] \frac{\sigma^{-1}(t) \gamma(t)}{e^T \sigma^{-1}(t) \gamma(t)}$$

$$+ \left[ A_1(t) \left( e^T \sigma^{-1}(t) \delta(t) \right) \right] \frac{\sigma^{-1}(t) \delta(t)}{e^T \sigma^{-1}(t) \delta(t)}$$

$$+ \frac{A_1(t)}{2} \sigma^{-1}(t) \text{diag}(\sigma(t))$$

$$+ A_2(t) \frac{\sigma^{-1}(t) e}{e^T \sigma^{-1}(t) e}$$

- High expected log price growth rate of return portfolio
- High cash flow rate of return portfolio
- Combination of rule-based portfolios
- Global minimum variance portfolio
Main Lessons

- Markowitz optimal portfolios naturally contain rule-based building blocks

- Decomposition of asset expected returns based on Ito’s lemma leads to following conclusions:
  - Know how to forecast asset price growth
  - Don’t know how to forecast asset price growth
    - optimal solution = blend of rule-based building blocks and traditional mean-variance portfolio
    - optimal solution = blend of rule-based building blocks only

- Optimal solution suggests diversification of four types of rule-based portfolio construction methodologies:
  - High cash flow/high dividend
  - Inverse volatility
  - Minimum variance
  - Equal weighting
The Connection between SPT and Rule-Based Investing:
Explaining the Long Term Outperformance of Rule-Based Portfolios vs. Cap-Weighted Benchmarks
Long Term Portfolio Returns in SPT

Log returns add together over time

Compute portfolio log returns (use Ito’s lemma)

\[
d \log Z_\pi(t) = \sum_{i=1}^{n} \pi_i(t) \left[ \gamma_i(t) + \delta_i(t) \right] dt + \frac{1}{2} \left[ \sum_{i=1}^{n} \pi_i(t) \sigma_{ii}(t) - \sum_{i,j=1}^{n} \sigma_{ij}(t) \pi_i(t) \pi_j(t) \right] dt + \sum_{i=1}^{n} \pi_i(t) \xi_i(t) dW(t)
\]

Portfolio log return ≠ weighted sum of asset log returns

Extra term = excess growth rate \( \gamma_{\pi}^*(t) \)

\[
d \log Z_\pi(t) = \sum_{i=1}^{n} \pi_i(t) d \log X_i(t) + \gamma_{\pi}^*(t) dt
\]
Key Invariance Property of Excess Growth Rate

- Excess growth rate is invariant whether computed with absolute returns or with returns calculated relative to any reference portfolio.

\[
\gamma^{*}_\pi(t) = \frac{1}{2} \left[ \sum_{i=1}^{n} \pi_i(t) \sigma_{ii}(t) - \sum_{i,j=1}^{n} \sigma_{ij}(t) \pi_i(t) \pi_j(t) \right] = \\
\frac{1}{2} \left[ \sum_{i=1}^{n} \pi_i(t) \tau_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t) \pi_j(t) \tau_{ij}(t) \right]
\]

- Asset tracking variances w.r.t reference portfolio
- Tracking variance risk w.r.t. reference portfolio

- Take case of portfolio = reference portfolio

\[
\gamma^{*}_\pi(t) = \frac{1}{2} \sum_{i=1}^{n} \pi_i(t) \tau_{ii}(t) \quad \text{Always positive, for long only portfolio}
\]
Long Term Portfolio Return Relative to the Market

Portfolio return relative to market portfolio

\[
d \log \left( \frac{Z_\pi(t)}{Z_\mu(t)} \right) = \sum_{i=1}^{n} \pi_i(t) d \log(\mu_i(t)) + \sum_{i=1}^{n} (\pi_i(t) - \mu_i(t)) \delta(t) dt + \gamma^*(t) dt
\]

Excess growth rate > 0

Differential cash flow rate of return

Stochastic market weight dynamics. Accounts for asset vs. market log growth rates
Market Diversity

- No single asset dominates the market in finite time
- Concept implicit in anti-trust laws
- The market portfolio holds all asset (including the largest cap.)
- Diversity implies: largest cap cannot be largest growth asset indefinitely
- The market portfolio is diverse if at no time a single asset accounts for almost the entire market capitalization. Mathematically, the market portfolio is diverse, if there exists a number $\kappa > 0$

\[ \mu_{\text{max}}(t) \leq 1 - \kappa, \quad t \in [0, \infty) \]

- Portfolio weights independent of market cap weights:

\[ \sum_{i=1}^{n} \pi_i(t) d \log(\mu_i(t)) = d \log \left[ \prod_{i=1}^{n} \mu_i(t)^{\pi_i(t)} \right] \]

Remains bounded if diversity is enforced and if no default occurs
Outperformance of Rule-Based Allocations vs. Cap-Weighted Benchmark: Explanation

- Any rule-based allocation scheme has positive excess growth rate.
- Any rule-based allocation scheme, if its cash flow rate of return is not significantly lower than the market’s, in the long term will out-perform the cap-weighted benchmark.

### U.S. 1964–2012

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Return</th>
<th>Standard Deviation</th>
<th>Sharpe Ratio</th>
<th>Information Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility Wt(^1)</td>
<td>12.2%</td>
<td>19.1%</td>
<td>0.36</td>
<td>0.34</td>
</tr>
<tr>
<td>Market Beta Wt(^2)</td>
<td>11.9%</td>
<td>19.8%</td>
<td>0.34</td>
<td>0.29</td>
</tr>
<tr>
<td>Downside Semi-Deviation Wt(^3)</td>
<td>12.1%</td>
<td>18.9%</td>
<td>0.37</td>
<td>0.36</td>
</tr>
<tr>
<td>U.S. Cap Wt(^4)</td>
<td>9.7%</td>
<td>15.3%</td>
<td>0.29</td>
<td>0.00</td>
</tr>
</tbody>
</table>

### What about inverse strategies?

| Inverse of Volatility Wt\(^1\) | 12.5%  | 15.6%              | 0.47         | 0.53              |
| Inverse of Market Beta Wt\(^2\)| 13.5%  | 15.0%              | 0.55         | 0.53              |
| Inverse of Downside Semi-Deviation Wt\(^3\)| 12.4%  | 15.6%              | 0.46         | 0.53              |
| U.S. Cap Wt\(^4\)              | 9.7%   | 15.3%              | 0.29         | 0.00              |

**Upside-down strategies also outperform … by a bigger margin!**

Source: Research Affiliates, LLC.
Outperformance of Rule-Based Allocations vs. Cap-Weighted Benchmark: Explanation

<table>
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<tbody>
<tr>
<td></td>
<td>Return</td>
</tr>
<tr>
<td>Book Value Wt$^5$</td>
<td>11.2%</td>
</tr>
<tr>
<td>5-Yr Avg Earnings Wt$^6$</td>
<td>11.2%</td>
</tr>
<tr>
<td>Fundamentals Wt$^7$</td>
<td>11.6%</td>
</tr>
<tr>
<td>Earnings Growth Wt$^8$</td>
<td>12.4%</td>
</tr>
<tr>
<td>U.S. Cap Wt$^4$</td>
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<tr>
<td>Inverse of 5-Yr Avg Earnings Wt$^6$</td>
<td>14.4%</td>
</tr>
<tr>
<td>Inverse of Fundamentals Wt$^7$</td>
<td>14.1%</td>
</tr>
<tr>
<td>Inverse of Earnings Growth Wt$^8$</td>
<td>10.3%</td>
</tr>
<tr>
<td>U.S. Cap Wt$^4$</td>
<td>9.7%</td>
</tr>
</tbody>
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They outperform, again.

Source: Research Affiliates, LLC.
The Connection between SPT and Rule-Based Investing:
The Relative Optimization Problem vs. Cap-Weighted Benchmarks:
Results and Empirical Evidence
Relative Portfolio Optimization

- Alpha strategy based on relative return between portfolio and benchmark

\[
\frac{dZ_{\pi}(t)}{Z_{\pi}(t)} - \frac{dZ_{\mu}(t)}{Z_{\mu}(t)} = d\log \left( \frac{Z_{\pi}(t)}{Z_{\mu}(t)} \right) + \frac{1}{2} \left[ \pi(t)^{\top} \sigma(t) \pi(t) - \frac{1}{2} \mu(t)^{\top} \sigma(t) \mu(t) \right] dt
\]

- Relative return = Bounded Noise + Drift + Mixed (Noise/Drift)Term

\[
\frac{dZ_{\pi}(t)}{Z_{\pi}(t)} - \frac{dZ_{\mu}(t)}{Z_{\mu}(t)} = \sum_{i=1}^{n} \pi_i(t) d\log(\mu_i(t)) + \sum_{i=1}^{n} (\pi_i(t) - \mu_i(t)) \delta_i(t) dt + \frac{1}{2} \left[ \pi(t)^{\top} \text{diag}(\sigma(t)) - \mu(t)^{\top} \sigma(t) \mu(t) \right] dt
\]

- Suppose that the portfolio deviates from the market portfolio by a set of market independent L/S tilts

\[
\pi_i(t) = \mu_i(t) + \eta_i(t)
\]
Relative Portfolio Optimization (cont’d)

Under this choice of portfolio weights, relative return depends only on tilt portfolio

\[
\frac{dZ_\pi(t)}{Z_\pi(t)} - \frac{dZ_\mu(t)}{Z_\mu(t)} = \sum_{i=1}^{n} \eta_i(t) d\log(\mu_i(t)) + \sum_{i=1}^{n} \eta_i(t) \delta_i(t) dt + \frac{1}{2} \eta(t)^T \text{diag}(\sigma(t)) dt
\]

Noise term bounded if diversity enforced. Maximize drift at fixed volatility. Optimal tilt portfolio = blend of “rule-based” building blocks only (HCF, IV, EW, GMV)

\[
\tilde{U} (\eta(t), \delta(t), \sigma(t)) = \eta(t)^T \delta(t) + \frac{1}{2} \eta(t)^T \text{diag}(\sigma(t)) - \lambda_1 \eta^T(t)e - \lambda_2 \left[ \eta(t)^T \sigma(t) \eta(t) - \chi^2 \right]
\]

Budget constraint: tilt portfolio weights sum to zero

Variance constraint: fix variance of tilt portfolio
Conclusions

- Some rule-based building blocks appear naturally in the mean-variance problem solution: global minimum variance (known result) + inverse volatility (new result)

- SPT decomposition of asset returns provides a simple explanation why Markowitz-optimal portfolios should include rule-based components complementing the mean-variance solution

- SPT also explains why (in the long term) almost any rule-based portfolio should outperform a cap-weighted benchmark in a diverse market (including Arnott’s inverse strategies….)

- SPT-optimal portfolios, maximizing relative drift at fixed tracking error vs. a cap-weighted benchmark, are always combinations of the market-capitalization-weighted index itself, and of four rule-based allocation schemes: GMV, EW, RP (inverse vol) and HCF

- SPT-optimal portfolios should be superior to individual rule-based allocations. Preliminary empirical work encouraging (although not yet fully satisfactory). More analyses currently in progress
Appendix
Empirical Check: US Equities vs. US Bonds

- Two asset classes: US Equities (MSCI USA Index) and US Bonds (BofA ML Corporate & Government Master Index)
- Total return monthly data since March 1976
- Estimate every month market weight evolution from total return index levels (starting from pre-defined initial allocation)
  \[ \hat{\mu}_i(t) = \frac{\hat{X}_i(t)}{\sum_{j=1}^{N} \hat{X}_j(t)} \]
- Total return index levels include contribution from dividends and coupons (part of market weight dynamics)
- Evaluate long term evolution of the maximum drift alpha strategy (monthly rebalanced) at different levels of risk
- If theory is right, optimal alpha strategy should be more efficient than any other alpha strategy based on individual rule-based allocations
Empirical Check: US Equities vs. US Bonds

Annualized Alpha Volatility

- Maximum Drift Frontier
- Alpha Strategy EW Portfolio
- Alpha Strategy RP Portfolio
- Alpha Strategy MDIV Portfolio